

# A Tail of a Quark in $\mathcal{N} = 4$ SYM

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## Abstract

We study the dynamics of a ‘composite’ or ‘dressed’ quark in strongly-coupled large- $N_c$   $\mathcal{N} = 4$  super-Yang-Mills, making use of the AdS/CFT correspondence. We show that the standard string dynamics nicely captures the physics of the quark and its surrounding non-Abelian field configuration, making it possible to derive a relativistic equation of motion that incorporates the effects of radiation damping. From this equation one can deduce a non-standard dispersion relation for the composite quark, as well as a Lorentz covariant formula for its rate of radiation. We explore the consequences of the equation in a few simple examples.

## 1 Introduction and Summary

When a charge radiates, energy conservation dictates that it must be subjected to a reactive force originating from its self-field, that tends to damp its motion. In the context of classical electrodynamics, the study of this damping or radiation reaction force began over a century ago [1, 2, 3, 4], and continues to this day [5, 6]. Reviews and additional references on the subject may be found in [7, 8, 9, 10].

In a non-relativistic approximation, the dynamics of an electron that is modeled as a vanishingly small spherically symmetric charge distribution is controlled by the Abraham-Lorentz equation [1, 2]

$$m \left( \ddot{\vec{x}} - t_e \dddot{\vec{x}} \right) = \vec{F} , \quad (1)$$

where  $\dot{\phantom{x}} \equiv d/dt$  and  $t_e \equiv 2e^2/3mc^3$  is a timescale set by the classical electron radius. In this equation, the damping force (the second term in the left-hand side) is seen to

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be proportional to the jerk  $\vec{j} \equiv \dot{\vec{a}} \equiv \ddot{\vec{v}}$ . The search for a Lorentz-covariant version of (1) led to the (Abraham-)Lorentz-Dirac equation [4],

$$m \left( \ddot{x}^\mu - t_e \left[ \ddot{x}^\mu - \frac{1}{c^2} \ddot{x}_\nu \ddot{x}^\nu \ddot{x}^\mu \right] \right) = \mathcal{F}^\mu, \quad (2)$$

with  $\dot{\phantom{x}} \equiv d/d\tau$ ,  $\tau$  the proper time (defined such that  $\dot{x}^\mu \dot{x}_\mu = -c^2$ ) and  $\mathcal{F}^\mu \equiv \gamma(\vec{F} \cdot \vec{v}/c, \vec{F})$  the four-force. The second term within the square brackets (proportional to the square of the proper acceleration) is the negative of the rate at which energy and momentum is carried away from the charge by radiation, according to the covariant Lienard(-Larmor-Heaviside-Abraham) formula. So, strictly speaking, it is only this term that can properly be called radiation reaction. The first term within the square brackets, usually called the Schott term, and whose spatial part yields the damping force of (1) in the non-relativistic limit, is known to arise from the effect of the charge's ‘near’ or ‘bound’ (as opposed to radiation) field [11, 7].

The appearance of third-order terms in (1) and (2) leads to unphysical behavior, including pre-accelerating and self-accelerating (or ‘runaway’) solutions. These deficiencies are known to originate from the assumption that the charge is pointlike.<sup>1</sup> For a charge distribution of small but finite size  $l$ , the above equations can be shown to be truncations of expressions that involve an infinite number of derivatives  $(ld/dt)^n$  (and generally include terms that are non-linear in these derivatives), but are physically sound as long as  $l > ct_e$  and [7, 8, 9].

Of course, one should keep in mind that the unphysical behavior implied by (1) and (2) would be visible only for time and distance scales smaller than the Compton wavelength  $\lambda_C \equiv \hbar/m$  of the charge, and thus lies outside of the actual range of validity of classical electrodynamics. How the preceding story generalizes to the case of fully quantum electrodynamics (QED) has been studied from different angles in [12, 13, 14, 15] and references therein. In particular, in [12] it was shown that, for a pointlike non-relativistic electron, QED leads to an equation of motion with an infinite number of higher derivatives, implying that the electron acquires an effective size  $l = \lambda_C$  due to its surrounding cloud of virtual particles.

Going further to non-Abelian gauge theories is a serious challenge.<sup>2</sup> paper to show that the AdS/CFT correspondence [18, 19, 20] allows us to examine this question rather easily in quantum strongly-coupled non-Abelian gauge theories. The essence of the matter is that in the context of this duality the quark corresponds to the tip of a string, whose body codifies the profile of the non-Abelian (near and radiation) fields sourced by the quark. In other words, the quark has a tail, and it is this tail that is responsible for the damping force. Indeed, this mechanism has already been seen at work in the recent computations of the drag force exerted on the quark by a

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<sup>1</sup>This assumption leads to the further complication of an infinite electromagnetic self-energy, which has already been absorbed within the renormalized mass  $m$  shown in (1) and (2). An alternative approach which circumvents this divergence has been proposed recently in [5].

<sup>2</sup>See respectively [16] and [17] for work on radiation within classical and (weakly-coupled) quantum Yang-Mills theory.

thermal plasma, which is described in dual language in terms of a string living on a black hole geometry [21, 22]. Our analysis makes it clear that, irrespective of whether a spacetime black hole is present or not, the body of the string plays the role of an energy sink, as befits its identification as the embodiment of the gluonic degrees of freedom.<sup>3</sup>

We expect this basic story to apply generally to all examples of the gauge/string duality, including cases with finite temperature or chemical potentials, but for simplicity we will concentrate on the case of quark motion in the vacuum of  $\mathcal{N} = 4$  super-Yang-Mills (SYM), where, building upon previous work [30, 25], we can achieve full analytic control. As has been remarked several times in the past, it is interesting that even in this non-confining theory the gluonic field configuration can be encoded in a ‘QCD’ string, albeit one that lives in a curved higher dimensional spacetime. Our results are a direct consequence of this amazing fact.

The paper is organized as follows. In Section 2, we explain how the standard string dynamics is mapped by the AdS/CFT correspondence onto the dynamics of an extended radiating particle. We begin by setting up our problem in Section 2.1, reviewing the relevant context and emphasizing some key features—most notably, the fact that the quark with finite mass that the correspondence puts at our disposal is automatically ‘dressed’ or ‘composite’, as discussed around Eq. (9). We then attack the problem in Section 2.2, where we derive an equation of motion for this quark, Eq. (28), which constitutes our main result.<sup>4</sup> As explained at length in the paragraphs that follow it, this equation is a nonlinear generalization of (2), which incorporates the effects of radiation damping, but has no pre-accelerating or self-accelerating solutions. From this equation one can read off a non-standard dispersion relation for the quark, Eq. (34), as well as a Lorentz covariant formula for its radiation rate, Eq. (35). An interesting novel feature in these expressions is their dependence on the external force exerted on the quark, which is a reflection of its extended, and hence deformable, nature. We close Section 2 by commenting on our failure to rewrite (28) in terms of an action principle.

In Section 3, we explore some of the physics implied by (28), specializing to a few simple examples. For the case of one-dimensional motion, we show in Section 3.1 that even though, as expected on physical grounds, zero external force implies zero acceleration, the converse is not true: the quark will not accelerate when subjected to an external force that takes the specific form (41) (which is identically zero only when the parameter  $t_0 \rightarrow \pm\infty$ ). More generally, for each given quark trajectory there is a one-parameter family of possible external forces. This again is a manifestation of the fact that, because of the extended character of the quark, the energy supplied to it can not only increase its velocity, but also modify its associated gluonic field profile. We end the paper by studying the nonrelativistic limit of (28) in Section 3.2,

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<sup>3</sup>On the other hand, energy loss via the string does turn out to be closely associated with the appearance of a *worldsheet* horizon, as noticed initially in [23, 24] at finite temperature and emphasized in [25] for the zero temperature case. This association has been further studied in [26, 27, 28, 29].

<sup>4</sup>A brief report of this derivation was given in the recent letter [31].

where the linearized form of the expressions allows us to make direct contact with the energy analysis of ([30, 25]) and to easily write down an action principle.

All in all, then, we have in (28) a physically sensible and interesting description of the dynamics of a composite quark in  $\mathcal{N} = 4$  SYM. This result serves, on the one hand, to illustrate the power of the AdS/CFT correspondence, and on the other, to shed some light on the largely uncharted terrain of radiation in strongly-coupled non-Abelian gauge theories. It would be interesting to extend this analysis in a number of directions. In particular, it seems worthwhile to explore the manner in which the split between intrinsic and radiated energy (and momentum) of the quark achieved in [30, 25] and the present paper, via examination of the string worldsheet, manifests itself in the gluonic field profile, by directly computing the expectation value of the energy-momentum tensor or similar local operators [32]. It is also natural to try to carry over some of the present methods to the finite temperature context [33], where one should be able to make contact with previous AdS/CFT analyses of energy loss in a thermal plasma (a rather large body of work detonated by the seminal works [21, 22, 34, 35]), including the interesting recent studies of Brownian motion [36, 37, 38].

## 2 From Strings to Quarks

### 2.1 Basic setup

It is by now well-known that strongly-coupled  $\mathcal{N} = 4$   $SU(N_c)$  SYM with coupling  $g_{YM}$  is dual to Type IIB string theory on a background that asymptotically approaches the  $AdS_5 \times S^5$  geometry<sup>5</sup>

$$\begin{aligned} ds^2 &= G_{MN} dx^M dx^N = \frac{R^2}{z^2} (-dt^2 + d\vec{x}^2 + dz^2) + R^2 d\Omega_5, \\ \frac{R^4}{l_s^4} &= g_{YM}^2 N_c \equiv \lambda \end{aligned} \quad (3)$$

(with a constant dilaton and  $N_c$  units of Ramond-Ramond five-form flux through the five-sphere), where  $l_s$  denotes the string length [18]. The radial direction  $z$  is mapped holographically into a variable length scale in the gauge theory [39]. The directions  $x^\mu \equiv (t, \vec{x})$  are parallel to the AdS boundary  $z = 0$  and are directly identified with the gauge theory directions. The state of IIB string theory described by the unperturbed metric (3) corresponds to the vacuum of the  $\mathcal{N} = 4$  SYM theory, and the closed string sector describing (small or large) fluctuations on top of it fully captures the gluonic (+ *adjoint* scalar and fermionic) physics.

From the gauge theory perspective, the introduction of an open string sector associated with a stack of  $N_f$  D7-branes in the geometry (3) is equivalent to the addition of  $N_f$  hypermultiplets in the *fundamental* representation of the  $SU(N_c)$

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<sup>5</sup>From this point on we work in natural units  $c = 1 = \hbar$ .

gauge group, breaking the supersymmetry down to  $\mathcal{N} = 2$  [40]. These are the degrees of freedom that we refer to as ‘quarks,’ even though they include both spin 1/2 and spin 0 fields. For  $N_f \ll N_c$ , the backreaction of the D7-branes on the geometry can be sensibly neglected; in the field theory this corresponds to working in a ‘quenched’ approximation which disregards quark loops (as well as the positive beta function they would generate). The D7-branes cover the four gauge theory directions, and extend along the radial AdS direction up from the boundary at  $z = 0$  to a position  $z = z_m$  where they ‘end’ (meaning that the  $\mathbf{S}^3 \subset \mathbf{S}^5$  that they are wrapped on shrinks down to zero size), which is inversely proportional to the quark mass,

$$z_m = \frac{\sqrt{\lambda}}{2\pi m} . \quad (4)$$

An isolated quark is dual to an open string that extends radially from the D7-branes to the AdS horizon at  $z \rightarrow \infty$ . The string dynamics follows as usual from the Nambu-Goto action

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det g_{ab}} \equiv \int d^2\sigma \mathcal{L}_{\text{NG}} , \quad (5)$$

where  $g_{ab} \equiv \partial_a X^M \partial_b X^N G_{MN}(X)$  ( $a, b = 0, 1$ ) denotes the induced metric on the worldsheet. In our work the entire string will be taken (consistently with the corresponding equations of motion) to lie at the ‘North Pole’ of the  $\mathbf{S}^5$  (the point where the  $\mathbf{S}^3 \subset \mathbf{S}^5$  that the D7-branes are wrapped on collapses to zero size), so the angular components of the metric (which are associated with the orientation of the gauge theory fields in the internal  $SU(4)$  symmetry group) will not play any role, and the lower endpoint of the string will necessarily lie at  $z = z_m$ .

We can exert an external force  $\vec{F}$  on the string endpoint by turning on an electric field  $F_{0i} = F_i$  on the D7-branes. This amounts to adding to the Nambu-Goto action the usual minimal coupling

$$S_{\text{F}} = \int d\tau A_\mu(X(\tau, z_m)) \partial_\tau X^\mu(\tau, z_m) ,$$

or, in terms of the quark worldline,

$$S_{\text{F}} = \int d\tau A_\mu(x(\tau)) \dot{x}^\mu(\tau) . \quad (6)$$

Notice that the string is being described (as is customary) in first-quantized language, and, as long as it is sufficiently heavy, we are allowed to treat it semiclassically. In gauge theory language, then, we are coupling a first-quantized quark to the gluonic (+ other SYM) field(s), and then carrying out the full path integral over the strongly-coupled field(s) (the result of which is codified by the AdS spacetime), but treating the path integral over the quark trajectory  $x^\mu(\tau)$  in a saddle-point approximation.

Variation of the string action  $S_{\text{NG}} + S_{\text{F}}$  implies the standard Nambu-Goto equation of motion for all interior points of the string, plus the standard boundary condition [41]

$$\Pi_{\mu}^z(\tau)|_{z=z_m} = \mathcal{F}_{\mu}(\tau) \quad \forall \tau, \quad (7)$$

where

$$\Pi_{\mu}^z \equiv \frac{\partial \mathcal{L}_{\text{NG}}}{\partial(\partial_z X^{\mu})} = \frac{\sqrt{\lambda}}{2\pi} \left( \frac{(\partial_{\tau} X)^2 \partial_z X_{\mu} - (\partial_{\tau} X \cdot \partial_z X) \partial_{\tau} X_{\mu}}{z^2 \sqrt{(\partial_{\tau} X \cdot \partial_z X)^2 - (\partial_{\tau} X)^2 (1 + (\partial_z X)^2)}} \right) \quad (8)$$

is the worldsheet (Noether) current associated with spacetime momentum, and we have recognized  $\mathcal{F}_{\mu} = -F_{\nu\mu} \partial_{\tau} x^{\nu} = (-\gamma \vec{F} \cdot \vec{v}, \gamma \vec{F})$  as the Lorentz four-force.

For the interpretation of our results it will be crucial to keep in mind that the quark described by this string is not ‘bare’ but ‘composite’ or ‘dressed’. This can be seen most clearly by working out the expectation value of the gluonic field surrounding a static quark located at the origin<sup>6</sup> [43],

$$\frac{1}{4g_{YM}^2} \langle \text{Tr } F^2(x) \rangle = \frac{\sqrt{\lambda}}{16\pi^2 |\vec{x}|^4} \left[ 1 - \frac{1 + \frac{5}{2} \left( \frac{2\pi m |\vec{x}|}{\sqrt{\lambda}} \right)^2}{\left( 1 + \left( \frac{2\pi m |\vec{x}|}{\sqrt{\lambda}} \right)^2 \right)^{5/2}} \right]. \quad (9)$$

For  $m \rightarrow \infty$  ( $z_m \rightarrow 0$ ), this is just the Coulombic field expected (by conformal invariance) for a pointlike charge. For finite  $m$  the profile is still Coulombic far away from the origin but in fact becomes non-singular at the location of the quark,

$$\frac{1}{4g_{YM}^2} \langle \text{Tr } F^2(x) \rangle = \frac{\sqrt{\lambda}}{128\pi^2} \left[ 15 \left( \frac{2\pi m}{\sqrt{\lambda}} \right)^4 - \frac{35}{|\vec{x}|^4} \left( \frac{2\pi m |\vec{x}|}{\sqrt{\lambda}} \right)^6 + \dots \right] \quad \text{for } |\vec{x}| < \frac{\sqrt{\lambda}}{2\pi m}.$$

As seen in these equations, the characteristic thickness of this non-Abelian charge distribution is precisely the length scale  $z_m$  defined in (4). This is then the size of the gluonic cloud that surrounds the quark, or in other words, the analog of the Compton wavelength for our non-Abelian source.

It is interesting to note that the string can be alternatively viewed as a Born-Infeld string, i.e., a soliton of the gauge and scalar fields on the D7-brane [44]. Since the small fluctuations of these fields (corresponding to microscopic open strings) are known to be dual to mesons, the composite quark itself can be thought of as a soliton constructed by aligning a large number of mesons [45]. The cloud surrounding our quark is then best thought of as ‘mesonic’ rather than ‘gluonic’. Mesons are indeed known to be the lightest states in the spectrum of the strongly-coupled gauge theory, with masses of order  $m_{\text{mes}} \equiv 1/z_m = 2\pi m/\sqrt{\lambda} \ll m$  [46], and form factors with size set by  $z_m$  [47].

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<sup>6</sup>More precisely, the operator in the left-hand side of (9) is the dual of the dilaton field, and includes not only the standard Yang-Mills term but also scalar and fermion contributions that can be found in [42], and which we suppress for notational simplicity.

So, to summarize,  $z_m$  can properly be called the *quark* Compton wavelength insofar as it gives the size of the cloud of virtual particles surrounding the quark, but one should bear in mind that it is given not by  $1/m$  but by  $1/m_{\text{mes}}$ , and in this sense it could also be referred to as the *meson* Compton wavelength.

## 2.2 Equation of motion for the quark

The first analysis of an accelerating quark via the AdS/CFT correspondence was carried out in [48], which used tools developed in [49] to study the dilatonic waves given off by small fluctuations on a radial string in  $\text{AdS}_5$ , and infer from them the profile of the gluonic field  $\langle \text{Tr } F^2(x) \rangle$  in the presence of a quark undergoing small oscillations. The results of [48] painted an interesting picture of the propagation of nonlinear waves in  $\mathcal{N} = 4$  SYM, but did not allow a definite identification of waves with the  $1/|\vec{x}|$  falloff associated with radiation. (More recently, this falloff has been successfully detected in the same setup through a calculation of the energy-momentum tensor  $\langle T_{\mu\nu} \rangle$  [50].)

In [48] it was noted that for  $z_m \rightarrow 0$  and in the linearized approximation, the string action (5) correctly implies the expected action for an ordinary non-relativistic particle of mass  $m \rightarrow \infty$ . In the present section we will obtain the relativistic generalization of this result retaining the full non-linear structure of the Nambu-Goto string, and then further extend the analysis to the case of finite  $m$ .

We will take as our starting point the results obtained in a remarkable paper by Mikhailov [30], which we now briefly review (a more detailed explanation can be found in [25]). This author considered an infinitely massive quark, and was able to solve the equation of motion for the dual string on  $\text{AdS}_5$ , for an *arbitrary* timelike trajectory of the string endpoint. In terms of the coordinates used in (3), his solution is

$$X^\mu(\tau, z) = z \frac{dx^\mu(\tau)}{d\tau} + x^\mu(\tau) , \quad (10)$$

with  $x^\mu(\tau)$  the worldline of the string endpoint at the AdS boundary— or, equivalently, the worldline of the dual, infinitely massive, quark— parametrized by its proper time  $\tau$ .

Combining (3) and (10), the induced metric on the worldsheet is found to be

$$g_{\tau\tau} = \frac{R^2}{z^2}(z^2\dot{x}^2 - 1), \quad g_{zz} = 0, \quad g_{z\tau} = -\frac{R^2}{z^2}, \quad (11)$$

implying in particular that the constant- $\tau$  lines are null, a fact that plays an important role in Mikhailov's construction. The solution (10) is 'retarded', in the sense that the behavior at time  $t = X^0(\tau, z)$  of the string segment located at radial position  $z$  is completely determined by the behavior of the string endpoint at an *earlier* time  $t_{\text{ret}}(t, z)$  obtained by projecting back toward the boundary along the null line at fixed  $\tau$ . An analogous 'advanced' solution built upon the same endpoint/quark trajectory can be obtained by reversing the sign of the first term in the right-hand side of (10).



In gauge theory language, this choice of sign corresponds to the choice between a purely outgoing or purely ingoing boundary condition for the waves in the gluonic field at spatial infinity. Both on the string and the gauge theory sides, more general configurations should of course exist, but obtaining them explicitly is difficult due to the highly non-linear character of the system. Henceforth we will focus solely on the retarded solutions, which are the ones that capture the physics of present interest, with influences propagating outward from the quark to infinity.

From the  $\mu = 0$  component of (10), parametrizing the quark worldline by  $x^0(\tau)$  instead of  $\tau$ , and using  $d\tau = \sqrt{1 - \vec{v}^2} dx^0$ , where  $\vec{v} \equiv d\vec{x}/dx^0$ , the relation that defines the retarded time follows as

$$t = z \frac{1}{\sqrt{1 - \vec{v}^2}} + t_{\text{ret}} , \quad (12)$$

where the endpoint velocity  $\vec{v}$  is meant to be evaluated at  $t_{\text{ret}}$ . In these same terms, the spatial components of (10) can be formulated as

$$\vec{X}(t, z) = z \frac{\vec{v}}{\sqrt{1 - \vec{v}^2}} + \vec{x}(t_{\text{ret}}) = (t - t_{\text{ret}})\vec{v} + \vec{x}(t_{\text{ret}}) . \quad (13)$$

Using (12) and (13), Mikhailov was able to rewrite the total string energy in the form

$$E(t) = \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^t dt_{\text{ret}} \frac{\vec{a}^2 - [\vec{v} \times \vec{a}]^2}{(1 - \vec{v}^2)^3} + E_q(\vec{v}(t)) , \quad (14)$$

where of course  $\vec{a} \equiv d\vec{v}/dx^0$ . The first term codifies the accumulated energy *lost* by the quark over all times prior to  $t$ , and is surprisingly seen to have precisely the same form as the standard Lienard formula from classical electrodynamics.<sup>7</sup> The second term in the above equation arises from a total derivative on the string worldsheet, and gives the expected Lorentz-covariant expression for the energy intrinsic to the quark [25],

$$E_q(\vec{v}) = \frac{\sqrt{\lambda}}{2\pi} \left( \frac{1}{\sqrt{1 - \vec{v}^2}} \frac{1}{z} \right) \Big|_{\infty}^{z_m=0} = \gamma m . \quad (15)$$

For the spatial momentum, [30, 25] similarly find

$$\vec{P}(t) = \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^t dt_{\text{ret}} \frac{\vec{a}^2 - [\vec{v} \times \vec{a}]^2}{(1 - \vec{v}^2)^3} \vec{v} + \vec{p}_q(\vec{v}(t)) , \quad (16)$$

with

$$\vec{p}_q = \frac{\sqrt{\lambda}}{2\pi} \left( \frac{\vec{v}}{\sqrt{1 - \vec{v}^2}} \frac{1}{z} \right) \Big|_{\infty}^{z_m=0} = \gamma m \vec{v} . \quad (17)$$

We see then that, in spite of the non-linear nature of the system, Mikhailov's procedure leads to a clean separation between the tip and the tail of the string, i.e.,

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<sup>7</sup>Possible experimental implications of this result have been explored in [51].



between the quark (including its near field) and its gluonic radiation field. We will now exploit this separation to study in more detail the dynamics of the quark.

Our initial observation is that, when we regard the Nambu-Goto action as a functional of the quark trajectory  $x^\mu$  by plugging (11) back into (5)+(6), we can explicitly carry out the integral over  $z$  to obtain

$$\begin{aligned} S_{\text{NG}} + S_{\text{F}} &= -\frac{R^2}{2\pi\alpha'} \int d\tau \int_{z_m \rightarrow 0}^{\infty} \frac{dz}{z^2} + \int d\tau A_\mu(x(\tau)) \dot{x}^\mu(\tau) \\ &= -m \int d\tau + \int d\tau A_\mu(x(\tau)) \dot{x}^\mu(\tau) , \end{aligned} \quad (18)$$

which is evidently the standard action for a pointlike externally forced relativistic particle (with mass  $m \rightarrow \infty$ ). Notice that the associated equation of motion does *not* include a damping force, which is just as one would expect for an infinitely massive charge, because the coefficient  $t_e \propto 1/m$  of the damping terms in (1) and (2) approaches zero as  $m \rightarrow \infty$ .

Let us now consider the more interesting case of a quark with finite mass,  $z_m > 0$ , where, as we emphasized in the previous subsection, our non-Abelian source is no longer pointlike but has size  $z_m$ . In this case the string endpoint is at  $z = z_m$ , and we must again require it to follow the given quark trajectory,  $x^\mu(\tau)$ . As before, this condition by itself does not pick out a unique string embedding. Just like we discussed for the infinitely massive case below (11), we additionally require the solution to be ‘retarded’ or ‘purely outgoing’, in order to focus on the gluonic field causally set up by the quark. As in [25], we can inherit this structure by truncating a suitably selected retarded Mikhailov solution. The embeddings of interest to us can thus be regarded as the  $z \geq z_m$  portions of the solutions (10), which are parametrized by data at the AdS boundary  $z = 0$ .<sup>8</sup> Henceforth we will use tildes to label these (now merely auxiliary) data, and distinguish them from the actual physical quantities (velocity, proper time, etc.) associated with the endpoint/quark at  $z = z_m$ , which will be denoted without tildes.

In this notation, (10) reads

$$X^\mu(\tilde{\tau}, z) = z \frac{d\tilde{x}^\mu(\tilde{\tau})}{d\tilde{\tau}} + \tilde{x}^\mu(\tilde{\tau}) . \quad (19)$$

Repeated differentiation of this equation with respect to  $\tilde{\tau}$  and evaluation at  $z = z_m$  (where we can read off the quark trajectory  $x^\mu(\tilde{\tau}) \equiv X^\mu(\tilde{\tau}, z_m)$ ) leads to the recursive

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<sup>8</sup>That this direct truncation indeed retains the retarded structure of the solutions is manifestly confirmed in our final rewriting (29), where information is seen to propagate upward along the body of the string, i.e., from the UV to the IR of the gauge theory.

relations

$$\begin{aligned}
\frac{dx^\mu}{d\tilde{\tau}} &= z_m \frac{d^2 \tilde{x}^\mu}{d\tilde{\tau}^2} + \frac{d\tilde{x}^\mu}{d\tilde{\tau}} , \\
\frac{d^2 x^\mu}{d\tilde{\tau}^2} &= z_m \frac{d^3 \tilde{x}^\mu}{d\tilde{\tau}^3} + \frac{d^2 \tilde{x}^\mu}{d\tilde{\tau}^2} , \\
&\vdots \\
\frac{d^n x^\mu}{d\tilde{\tau}^n} &= z_m \frac{d^{n+1} \tilde{x}^\mu}{d\tilde{\tau}^{n+1}} + \frac{d^n \tilde{x}^\mu}{d\tilde{\tau}^n} .
\end{aligned} \tag{20}$$

Adding these equations respectively multiplied by  $(-z_m)^{n-1}$ , we can deduce that

$$\frac{d\tilde{x}^\mu}{d\tilde{\tau}} = \frac{dx^\mu}{d\tilde{\tau}} - z_m \frac{d^2 x^\mu}{d\tilde{\tau}^2} + z_m^2 \frac{d^3 x^\mu}{d\tilde{\tau}^3} - \dots , \tag{21}$$

and, upon further differentiation,

$$\frac{d^2 \tilde{x}^\mu}{d\tilde{\tau}^2} = \frac{d^2 x^\mu}{d\tilde{\tau}^2} - z_m \frac{d^3 x^\mu}{d\tilde{\tau}^3} + z_m^2 \frac{d^4 x^\mu}{d\tilde{\tau}^4} - \dots . \tag{22}$$

This last expression already takes us halfway towards the equation we are after, but we still need to find a relation between  $d\tilde{\tau}$  and the endpoint/quark proper time  $d\tau$ , and similarly rewrite  $d^2 \tilde{x}^\mu / d\tilde{\tau}^2$  in terms of quantities at the actual string boundary  $z = z_m$  instead of the auxiliary data at  $z = 0$ .

The first task is easy: from (19) it follows that

$$dX^\mu = dz \frac{d\tilde{x}^\mu}{d\tilde{\tau}} + d\tilde{\tau} \left( z \frac{d^2 \tilde{x}^\mu}{d\tilde{\tau}^2} + \frac{d\tilde{x}^\mu}{d\tilde{\tau}} \right) ,$$

which evaluated at fixed  $z = z_m$  implies

$$dx^\mu = d\tilde{\tau} \left( z_m \frac{d^2 \tilde{x}^\mu}{d\tilde{\tau}^2} + \frac{d\tilde{x}^\mu}{d\tilde{\tau}} \right) ,$$

and therefore

$$d\tau^2 \equiv -dx^\mu dx_\mu = d\tilde{\tau}^2 \left[ 1 - z_m^2 \left( \frac{d^2 \tilde{x}}{d\tilde{\tau}^2} \right)^2 \right] . \tag{23}$$

To arrive at this last equation, we have made use of the fact that  $\tilde{\tau}$  is by definition the proper time for the auxiliary worldline at  $z = 0$ , so  $(d\tilde{x}/d\tilde{\tau})^2 = -1$  and  $(d\tilde{x}/d\tilde{\tau}) \cdot (d^2 \tilde{x}/d\tilde{\tau}^2) = 0$ .

What remains then is to express  $d^2 \tilde{x}^\mu / d\tilde{\tau}^2$  as a function of quark data. For this we note first that, upon substituting the solution (19), the momentum current (8) (with appropriate tildes) evaluated at  $z = z_m$  simplifies to

$$\frac{2\pi}{\sqrt{\lambda}} \tilde{\Pi}_\mu^z = \frac{1}{z_m} \frac{d^2 \tilde{x}_\mu}{d\tilde{\tau}^2} + \left( \frac{d^2 \tilde{x}}{d\tilde{\tau}^2} \right)^2 \frac{d\tilde{x}_\mu}{d\tilde{\tau}} . \tag{24}$$

To avoid possible confusion, we should note that the tilde in the left-hand side does not indicate evaluation at  $z = 0$  (as all other tildes do), but the fact that this current is defined as charge (momentum) flow per unit  $\tilde{\tau}$ . The corresponding flow per unit  $\tau$  is clearly just<sup>9</sup>  $\Pi_\mu^z = (\partial\tilde{\tau}/\partial\tau)\tilde{\Pi}_\mu^z$ , and it is this object which according to (7) must equal the external force  $\mathcal{F}_\mu$ . Using this, (23) and the first equation of (20) in (24), one can deduce (after some straightforward algebra) that

$$\frac{d^2\tilde{x}_\mu}{d\tilde{\tau}^2} = \frac{1}{\sqrt{1 - z_m^4 \mathcal{F}^2}} \left( z_m \mathcal{F}_\mu - z_m^3 \mathcal{F}^2 \frac{dx_\mu}{d\tau} \right), \quad (25)$$

where we have used the abbreviation  $\mathcal{F}_\mu \equiv (2\pi/\sqrt{\lambda})\mathcal{F}_\mu$ . Since  $\mathcal{F}^\mu dx_\mu/d\tau = 0$  (which is merely the statement that no work is done on the quark in its instantaneous rest frame), this implies that  $(d^2\tilde{x}/d\tilde{\tau}^2)^2 = z_m^2 \mathcal{F}^2$ , which allows (23) to be simplified into

$$d\tilde{\tau} = \frac{d\tau}{\sqrt{1 - z_m^4 \mathcal{F}^2}}. \quad (26)$$

Using (25) and (26), we can finally rewrite (22) in the form

$$\begin{aligned} \frac{z_m \mathcal{F}^\mu}{\sqrt{1 - z_m^4 \mathcal{F}^2}} &= \left( \frac{z_m^3 \mathcal{F}^2}{1 - z_m^4 \mathcal{F}^2} \right) \frac{dx^\mu}{d\tau} + \sqrt{1 - z_m^4 \mathcal{F}^2} \frac{d}{d\tau} \left[ \sqrt{1 - z_m^4 \mathcal{F}^2} \frac{dx^\mu}{d\tau} \right] \\ &\quad - z_m \sqrt{1 - z_m^4 \mathcal{F}^2} \frac{d}{d\tau} \left[ \sqrt{1 - z_m^4 \mathcal{F}^2} \frac{d}{d\tau} \left[ \sqrt{1 - z_m^4 \mathcal{F}^2} \frac{dx^\mu}{d\tau} \right] \right] + \dots \end{aligned} \quad (27)$$

This equation of motion for the quark contains an infinite number of derivatives of  $x^\mu$ , precisely as one would expect for an extended color charge distribution, based on the classical or quantum electrodynamic analogs [7, 8, 12] mentioned in the Introduction. Notice that to arrive at this result we have made no assumption about the profile of the charge distribution. The dual string dynamics automatically incorporates the physics of this profile, which is codified by the slope  $\vec{s} \equiv \partial_z \vec{X}(z_m, t)$  [25]. It would be interesting to explore this connection in more detail through a calculation of  $\langle \text{Tr } F^2(x) \rangle$  and similar observables for an accelerating quark in vacuum [48, 50, 32].

A more manageable form of the equation of motion can be obtained by going back to the second equation in (20),

$$\frac{d^2 x^\mu}{d\tilde{\tau}^2} = \frac{d^2 \tilde{x}^\mu}{d\tilde{\tau}^2} + z_m \frac{d^3 \tilde{x}^\mu}{d\tilde{\tau}^3}.$$

Through (25), (26) and (4), this can be reexpressed as

$$\frac{d}{d\tau} \left( \frac{m \frac{dx^\mu}{d\tau} - \frac{\sqrt{\lambda}}{2\pi m} \mathcal{F}^\mu}{\sqrt{1 - \frac{\lambda}{4\pi^2 m^4} \mathcal{F}^2}} \right) = \frac{\mathcal{F}^\mu - \frac{\sqrt{\lambda}}{2\pi m^2} \mathcal{F}^2 \frac{dx^\mu}{d\tau}}{1 - \frac{\lambda}{4\pi^2 m^4} \mathcal{F}^2}, \quad (28)$$

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<sup>9</sup>The transformation rules for  $\Pi_\mu^a$  under more general reparametrizations can be found in, e.g., [52].

which is the equation we advertised in the Introduction. Notice that it involves only the four-velocity and four-acceleration of the quark, so, in going from (27) to (28), we have traded an infinite number of higher derivatives for a somewhat more complicated  $\mathcal{F}^\mu$  dependence. This is to some extent analogous to the possibility of trading (1), (2) or its non-pointlike generalizations for an integro-differential (nonlocal in the force) equation with derivatives of  $x^\mu$  only up to second order [7, 8]. What is different is that our end result, equation (28), does not involve any nonlocality. (It is also highly nonlinear, because it includes effects that are neglected for simplicity in nearly all previous analyses of radiation damping.)

Before proceeding with the analysis of (28), we would like to note for future use that the information we have gathered in the process of its derivation, and more specifically, equations (20) and (25), allow Mikhailov's solution (10) to be rewritten purely in terms of  $z = z_m$  data as

$$X^\mu(\tau, z) = \left( \frac{z - z_m}{\sqrt{1 - z_m^4 \mathcal{F}^2}} \right) \left( \frac{dx^\mu}{d\tau} - z_m^2 \mathcal{F}^\mu \right) + x^\mu(\tau) . \quad (29)$$

A first check on (28) is to note that it correctly reduces to  $md^2x^\mu/d\tau^2 = \mathcal{F}^\mu$  in the pointlike limit  $m \rightarrow \infty$  (where the Compton wavelength  $z_m \rightarrow 0$ ). We will now perform a more substantial check by showing that it also makes firm contact with the results of [25] at finite  $m$ .

In [25], two of us generalized the analysis of Mikhailov [30] to the case of a quark with finite mass, concentrating for simplicity on motion along one dimension. We showed that under such circumstances the total string (= field + quark) energy  $E$  and momentum  $P$  are no longer given by (14) and (16), but become

$$\begin{aligned} E(t) &= \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^t dt \frac{F^2}{m^2} \left( \frac{1 - \frac{\sqrt{\lambda}}{2\pi m^2} Fv}{1 - \frac{\lambda}{4\pi^2 m^4} F^2} \right) + \frac{1 - \frac{\sqrt{\lambda}}{2\pi m^2} Fv}{\sqrt{1 - \frac{\lambda}{4\pi^2 m^4} F^2}} \gamma m , \\ P(t) &= \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^t dt \frac{F^2}{m^2} \left( \frac{v - \frac{\sqrt{\lambda}}{2\pi m^2} F}{1 - \frac{\lambda}{4\pi^2 m^4} F^2} \right) + \frac{v - \frac{\sqrt{\lambda}}{2\pi m^2} F}{\sqrt{1 - \frac{\lambda}{4\pi^2 m^4} F^2}} \gamma m , \end{aligned} \quad (30)$$

with  $F$  the external force. These expressions show that for  $m < \infty$  the rate (seen inside the integrals) at which energy/momentum is radiated by the quark differs from the Lienard result, and the formulas for the intrinsic energy  $E_q$  and momentum  $p_q$  of the quark (given by the terms that follow the integrals) are similarly non-standard. Now, the total momentum  $P$  of the string (= field + quark) changes only due to the force that we exert on the endpoint (= quark), so  $dP/dt = F$ , or, using (30),

$$\frac{\sqrt{\lambda}}{2\pi} \frac{F^2}{m^2} \left( \frac{v - \frac{\sqrt{\lambda}}{2\pi m^2} F}{1 - \frac{\lambda}{4\pi^2 m^4} F^2} \right) + \frac{d}{dt} \left( \frac{m\gamma v - \frac{\sqrt{\lambda}}{2\pi m} \gamma F}{\sqrt{1 - \frac{\lambda}{4\pi^2 m^4} F^2}} \right) = F . \quad (31)$$

Let us now compare this against our equation of motion (28). For linear motion along direction  $x^1$ , we have  $dx^\mu/d\tau = \gamma(1, v)$  and  $\mathcal{F}^\mu = \gamma(Fv, F)$ , so the  $\mu = 1$

component of (28) reads

$$\frac{d}{dt} \left( \frac{m\gamma v - \frac{\sqrt{\lambda}}{2\pi m} \gamma F}{\sqrt{1 - \frac{\lambda}{4\pi^2 m^4} F^2}} \right) = \frac{F - \frac{\sqrt{\lambda}}{2\pi m^2} F^2 v}{1 - \frac{\lambda}{4\pi^2 m^4} F^2} . \quad (32)$$

This is in precise agreement with (31). Similarly, it is easy to see that the  $\mu = 0$  component of (28) yields the expected result  $dE/dt = Fv$  with  $E$  as in (30).

We have thus verified that our quark equation of motion reproduces the energy/momentum split between quark and radiation field previously deduced in [25] for the case of one-dimensional motion. It becomes clear then that (28) encodes the covariant generalization of this split to the case of arbitrary motion— a result that would have been extraordinarily difficult to obtain using the non-covariant approach of [25]. To make this generalization explicit, we rewrite our equation in the form

$$\frac{dP^\mu}{d\tau} \equiv \frac{dp_q^\mu}{d\tau} + \frac{dP_{\text{rad}}^\mu}{d\tau} = \mathcal{F}^\mu, \quad (33)$$

recognizing

$$p_q^\mu = \frac{m \frac{dx^\mu}{d\tau} - \frac{\sqrt{\lambda}}{2\pi m} \mathcal{F}^\mu}{\sqrt{1 - \frac{\lambda}{4\pi^2 m^4} \mathcal{F}^2}} \quad (34)$$

as the intrinsic four-momentum of the quark, and

$$\frac{dP_{\text{rad}}^\mu}{d\tau} = \frac{\sqrt{\lambda} \mathcal{F}^2}{2\pi m^2} \left( \frac{\frac{dx^\mu}{d\tau} - \frac{\sqrt{\lambda}}{2\pi m^2} \mathcal{F}^\mu}{1 - \frac{\lambda}{4\pi^2 m^4} \mathcal{F}^2} \right) \quad (35)$$

as the rate at which four-momentum is carried away from the quark by chromo-electromagnetic radiation.

Using again the fact that  $\mathcal{F} \cdot \partial_\tau x = 0$ , we can immediately deduce from (34) the mass-shell condition  $p_q^2 = -m^2$ , which shows in particular that  $p_q^\mu$  is indeed a four-vector, and so the split  $P^\mu = p_q^\mu + P_{\text{rad}}^\mu$  defined in (33)-(35) is correctly Lorentz covariant. As we indicated above,  $P_{\text{rad}}^\mu$  represents the portion of the total four-momentum stored at any given time in the *purely radiative* part of the gluonic field set up by the quark. The remainder,  $p_q^\mu$ , includes the contribution of the *near* field sourced by our particle, or in quantum mechanical language, of the gluonic cloud surrounding the quark, which gives rise to the deformed dispersion relation seen in (34). In other words,  $p_q^\mu$  is the four-momentum of the ‘dressed’ or ‘composite’ quark. All of this is completely analogous to the classical electromagnetic case that we briefly reviewed in the Introduction, and in particular, to the covariant splitting of the Maxwell tensor achieved in [11]. It is truly remarkable that the AdS/CFT correspondence grants us such direct access to this piece of strongly-coupled non-Abelian physics.

Now that we have performed some checks on (28) and understood its proper physical interpretation, we should consider its implications. As noticed already in [25] for the case of linear motion, a salient feature of the equation of motion (28),

as well as the dispersion relation (34) and radiation rate (35), is the presence of a divergence when  $\mathcal{F}^2 = \mathcal{F}_{\text{crit}}^2$ , where

$$\mathcal{F}_{\text{crit}}^2 = \frac{4\pi^2 m^4}{\lambda} \quad (36)$$

is the critical value at which the force becomes strong enough to nucleate quark-antiquark pairs (or, in dual language, to create open strings) [24].

Let us now examine the behavior of a quark that is sufficiently heavy, or is forced sufficiently softly, that the condition  $\sqrt{\lambda|\mathcal{F}^2|}/2\pi m^2 \ll 1$  (i.e.,  $|\mathcal{F}^2| \ll |\mathcal{F}_{\text{crit}}^2|$ ) holds. It is then natural to expand the equation of motion in a power series in this small parameter. To zeroth order in this expansion, we have the pointlike result  $m d^2 x^\mu / d\tau^2 = \mathcal{F}^\mu$ , as we had already mentioned above. If we instead keep terms up to first order, we find

$$m \frac{d}{d\tau} \left( \frac{dx^\mu}{d\tau} - \frac{\sqrt{\lambda}}{2\pi m^2} \mathcal{F}^\mu \right) = \mathcal{F}^\mu - \frac{\sqrt{\lambda}}{2\pi m^2} \mathcal{F}^2 \frac{dx^\mu}{d\tau} . \quad (37)$$

In the  $\mathcal{O}(\sqrt{\lambda})$  terms it is consistent, to this order, to replace  $\mathcal{F}^\mu$  with its zeroth order value, thereby obtaining

$$m \left( \frac{d^2 x^\mu}{d\tau^2} - \frac{\sqrt{\lambda}}{2\pi m} \frac{d^3 x^\mu}{d\tau^3} \right) = \mathcal{F}^\mu - \frac{\sqrt{\lambda}}{2\pi} \frac{d^2 x^\nu}{d\tau^2} \frac{d^2 x_\nu}{d\tau^2} \frac{dx^\mu}{d\tau} . \quad (38)$$

Interestingly, (38) coincides *exactly* with the Lorentz-Dirac equation (2)! As expected from the discussion in the preceding paragraphs, on the left-hand side we find the Schott term (associated with the near field of the quark) arising from the modified dispersion relation (34). On the right-hand side we see the radiation reaction force given by the covariant Lienard formula, as expected from the result (14) [30], which is the pointlike limit of the radiation rate (35). Moreover, by comparing (2) and (38) we learn that it is  $z_m = \sqrt{\lambda}/2\pi m$  that plays the role of characteristic time/size  $t_e$  for the composite quark. As we discussed around (9),  $z_m$  is the Compton wavelength (i.e., size of the gluonic cloud) of the quark, which is indeed the natural quantum scale of the problem.

If we continue with the expansion of (28), the result at second order can be written as

$$m \ddot{x}^\mu - \frac{\sqrt{\lambda}}{2\pi} (\ddot{x}^\mu - \ddot{x}^\nu \ddot{x}_\nu \dot{x}^\mu) + \frac{\lambda}{4\pi^2 m} \left( \ddot{x}^\mu - 3 \ddot{x}^\nu \ddot{x}_\nu \ddot{x}^\mu - \frac{3}{2} \ddot{x}^\nu \ddot{x}_\nu \ddot{x}^\mu \right) = \mathcal{F}^\mu .$$

We can of course continue this expansion procedure to arbitrarily high order in  $\sqrt{\lambda|\mathcal{F}^2|}/2\pi m^2$ , and at order  $n$  in this parameter, we would obtain an equation with derivatives up to order  $n+2$ . Now, it is interesting to note that, in the case of non-relativistic QED, there are actually two different scales that appear in the corresponding equation of motion [12]: while the terms with derivatives of fourth and higher order are all characterized by the *quantum* (Compton) radius  $\lambda_C = \hbar/m$ , the third-derivative (i.e., Abraham-Lorentz) term involves only the *classical* electron radius  $ct_e = e^2/mc^2 \ll \lambda_C$ . In our strongly-coupled non-Abelian setting, the analogs

of these two scales happen to coincide. On the one hand,  $z_m$  is analogous to  $\lambda_C$  in that it gives the size of the cloud of virtual particles surrounding the quark, which as we explained at the end of Section 2, is set by the meson (and not the quark) mass,  $z_m = 1/m_{\text{mes}}$ . On the other hand,  $z_m$  is also analogous to  $ct_e$  in that it gives the radius of a charge distribution whose chromoelectrostatic energy equals the quark mass  $m$ , if we take into account the *strong-coupling* form of the potential  $V(L) \propto \sqrt{\lambda}/L$  [53].

As promised in the Introduction, our full equation (28) is thus recognized as an extension of the Lorentz-Dirac equation that automatically incorporates the size  $z_m$  of our non-classical, non-pointlike and non-Abelian source. The passage from (38) to (28), which can be viewed intuitively as the addition of an infinite number of higher derivative terms  $(z_m d/d\tau)^n$ , has a profound impact on the space of solutions. Here we will limit ourselves to two general observations, leaving the search for specific examples of solutions to the next section. The first is to notice that, unlike its classical electrodynamic counterparts (1) and (2), our composite quark equation of motion has no pre-accelerating or self-accelerating solutions. That is to say, the behavior of the quark at any given time  $\tau$  does not depend on  $\mathcal{F}^\mu(\tau')$  at  $\tau' > \tau$ , and, in the (continuous) absence of an external force, (28) uniquely predicts that the four-acceleration of the quark must vanish. Our second observation, however, is that the converse to this last statement is not true: constant four-velocity does not uniquely imply a vanishing force. We will expand on this in the examples of the next section.

Notice that, in the systematic approach from our result (28) to the Lorentz-Dirac equation, the pathologies that afflict the latter appear only in the very last step, when the  $\overset{\circ}{x}^\mu$  term is introduced upon approximating (37) by (38). Indeed, it has often been advocated to eliminate these pathologies by a ‘reduction of order’ procedure (see, e.g., [54, 5]), which treats the damping force as a perturbative correction and thereby justifies the replacement of (38) by (37). (A closely related line of reasoning can be found in [55].) It is therefore satisfying to see that the dressed quark equation of motion predicted by AdS/CFT, Eq. (28), automatically comes out in ‘reduced order’ form.

It is curious to note that (28), which incorporates the effect of radiation damping on the quark, has been obtained from (10), which does *not* include such damping for the string itself. The supergravity fields set up by the string are of order  $1/N_c^2$ , and therefore subleading at large  $N_c$ . Even more curious [48] is the fact that it is precisely these suppressed fields that encode the gluonic field profile generated by the quark, as has been explored in great detail (mostly at finite temperature) in recent years (see, e.g., [56] and references therein). It would be interesting to explore how the split into near and radiation fields is achieved from this perspective, but we leave this problem to future work [32].

It is natural to inquire whether the equation of motion (28) can be encoded in a variational principle. Since the complete system includes the composite quark in interaction with its radiation field, and the four-momentum of the latter is given by an integral over the quark worldline, at least naively we would expect that, if it is at all possible to write down an action, it should depend bilocally on the worldline.



This would rule out the obvious candidate action, namely  $S = S_{\text{NG}} + S_{\text{F}}$  with the classical solution (19) plugged in. Indeed, the latter procedure simply leads again to (18) (with  $\tau$  in the Nambu-Goto term relabeled to  $\tilde{\tau}$ ), which (23) allows to be converted into

$$S = -m \int \frac{d\tau}{\sqrt{1 - \frac{\lambda}{4\pi^2 m^4} \mathcal{F}^2}} + \int d\tau A_\mu(x(\tau)) \dot{x}^\mu(\tau) \quad (39)$$

Unlike what happened in the pointlike (and no radiation damping) limit  $m \rightarrow \infty$ , in this case variation of  $S$  with respect to  $x^\mu$  holding  $\mathcal{F}^\mu$  fixed does *not* yield the correct equation of motion (28). Of course,  $S$  is by definition the correct on-shell action for the system, in the sense that it yields the right number when evaluated on a given worldline (using the  $\mathcal{F}^\mu(\tau)$  obtained by solving (28)), but it would somehow need to be rewritten in bilocal form to explicitly show all relevant  $x^\mu$  dependence and thus constitute the desired variational principle.

## 3 Examples

### 3.1 One-dimensional motion

We have already seen in Section 2.2 that, in the case of one-dimensional motion, our general equation of motion (28) reduces to (32). The latter can be further simplified to

$$a = \frac{z_m \mathcal{F}(1 - v^2)^{3/2}}{\sqrt{1 - z_m^4 \mathcal{F}^2}} + \frac{z_m^2 \dot{\mathcal{F}}(1 - v^2)}{1 - z_m^4 \mathcal{F}^2} . \quad (40)$$

where we have preferred to express the prefactors in terms of the quark Compton wavelength  $z_m$  instead of its mass  $m$ , and as in (25) we have used the abbreviation  $\mathcal{F} \equiv (2\pi/\sqrt{\lambda})F$ . Here we see directly that, in contrast with the usual case, the acceleration at any given time depends not only on the value of the applied force but also on its rate of change.

It follows from (40) that a quark that is free for any extended period of time will not accelerate. In other words, as we had already indicated in the general discussion below (38), there are no self-accelerating solutions, which is just as one would have expected given the extended nature of the charge. On the other hand, the very fact that the charge has a ‘deformable’ internal structure implies that there is more than one way to get it to follow any given trajectory. Indeed, for any choice of  $v(t)$ , (40) fixes the applied force  $\mathcal{F}(t)$  not algebraically, but through a differential equation that inevitably gives rise to a one-parameter family of solutions (differing by their initial conditions).

The simplest example of this non-uniqueness of the external force is the case of constant velocity, where (40) can be easily seen to imply that

$$\mathcal{F}(t) = \pm \frac{1}{z_m^2} \text{sech} \left( \frac{\sqrt{1 - v^2}}{z_m} (t - t_0) \right) , \quad (41)$$

with  $t_0$  an integration constant. Only for  $t_0 \rightarrow \pm\infty$  does one recover the simple result  $F(t) = 0$ . For all finite values of the integration constant, the force starts out at zero at asymptotically early times, rises steadily until it attains the critical value  $F = \sqrt{\lambda}/2\pi z_m^2$  given by (36) at  $t = t_0$ , and then approaches zero again as  $t \rightarrow \infty$ .

It is certainly peculiar that one can continually apply a force to the quark and still have it move at constant velocity. Nonetheless, it is easy to verify through numerical integration that indeed the application of the force (41) to the endpoint of a string whose initial profile is chosen in accord with (29) produces no acceleration. The energy provided to the system by  $F(t)$  does not translate into an increase of the endpoint velocity, but into a continuous modification of the string tail, or, in gauge theory language, a change of the gluonic field profile. In fact, with the formulas derived in the previous section, we can make a much more precise statement: (34) and (35) reduce to [25]

$$E_q = \left( \frac{1 - z_m^2 \not{F} v}{\sqrt{1 - z_m^4 \not{F}^2}} \right) \gamma m, \quad \frac{dE_{\text{rad}}}{dt} = z_m^2 \not{F}^2 \left( \frac{1 - z_m^2 \not{F} v}{1 - z_m^4 \not{F}^2} \right),$$

where we see that application of the force (41) results in

$$\begin{aligned} E_q &= \left\{ \left| \coth \left( \frac{t - t_0}{\gamma z_m} \right) \right| \mp v \left| \text{csch} \left( \frac{t - t_0}{\gamma z_m} \right) \right| \right\} \gamma m, \\ \frac{dE_{\text{rad}}}{dt} &= \text{csch}^2 \left( \frac{t - t_0}{\gamma z_m} \right) \left[ 1 - v \text{sech}^2 \left( \frac{t - t_0}{\gamma z_m} \right) \right], \end{aligned}$$

which show precisely what fraction of the energy goes into (or comes out from) rearranging the near gluonic field, and what fraction goes into radiation. In the limit  $t_0 \rightarrow \pm\infty$ , no force is applied and we of course recover  $E_q = \gamma m$ ,  $dE_{\text{rad}}/dt = 0$  at all times. Similar conclusions can be drawn about the linear quark momentum.

Notice that the width of the time interval over which the force (41) differs appreciably from zero is just the Compton wavelength of the quark,  $z_m$  (with an appropriate Lorentz dilation factor), which serves to emphasize that this peculiar phenomenon is made possible only due to the non-pointlike nature of our non-Abelian source.

This non-uniqueness of the force is equally present for arbitrary trajectories. Another example that is easy to study is the case of constant force. If we assume that  $F(t) = \text{constant}$ , (40) reduces to  $d(\gamma v)/dt = z_m \not{F}/\sqrt{1 - z_m^4 \not{F}^2}$ , which has the same form as the standard equation of motion for a particle with constant proper acceleration, except that the force here is non-linearly rescaled. The solution is thus

$$x(t) = x_0 \pm \sqrt{\frac{1}{z_m^2 \not{F}^2} - z_m^2 + (t - t_0)^2},$$

where  $x_0$  and  $t_0$  are integration constants. But if we run the argument in reverse, and ask what type of force would lead to the hyperbolic motion  $x(t) = x_0 \pm \sqrt{C^2 + (t - t_0)^2}$ , we see again that the differential equation (40) admits solutions other than the obvious  $\not{F}(t) = 1/z_m \sqrt{C^2 + z_m^2}$ .

Before closing this subsection, we would like to make one additional observation. After some straightforward algebra, it is easy to see that (35) can be rewritten in the form

$$\frac{dP_{\text{rad}}^\mu}{d\tau} = \frac{\sqrt{\lambda}}{2\pi m^2} \left[ \frac{\mathcal{F}^2}{\sqrt{1 - \frac{\lambda}{4\pi^2 m^4} \mathcal{F}^2}} \right] p_q^\mu . \quad (42)$$

Plugging this into (33), one obtains a first order differential equation for  $p_q^\mu$  with the same structure as the Langevin equation, but with a friction coefficient that depends on the external force,

$$\frac{dp_q^\mu}{d\tau} = -\mu(\mathcal{F})p_q^\mu + \mathcal{F}^\mu , \quad (43)$$

where  $\mu(\mathcal{F}) \equiv (\sqrt{\lambda}/2\pi m^2)\mathcal{F}^2/\sqrt{1 - \lambda/4\pi^2 m^4 \mathcal{F}^2}$ . In the case of motion in one spatial dimension, this equation can be solved analytically,

$$p_q = \exp \left( -\frac{\sqrt{\lambda}}{2\pi m^2} \int^t \frac{\mathcal{F}^2(x)}{\sqrt{1 - \frac{\lambda}{4\pi^2 m^4} \mathcal{F}^2(x)}} dx \right) \times \left[ A + \int^t \exp \left( \frac{\sqrt{\lambda}}{2\pi m^2} \int^y \frac{\mathcal{F}^2(x)}{\sqrt{1 - \frac{\lambda}{4\pi^2 m^4} \mathcal{F}^2(x)}} dx \right) \mathcal{F}(y) dy \right] , \quad (44)$$

with  $A$  an integration constant. We should stress that this solution is valid only in the case of one dimensional motion, where  $\mathcal{F}^2 = \vec{F}^2$ . In two or three spatial dimensions we have not been able to construct an explicit solution, because in that case  $\mathcal{F}^2$  involves the velocity of the quark, and remains undetermined until we find the quark trajectory. Nevertheless, it is interesting to note that, in the case of constant  $\mathcal{F}^2$  (which is *not* the same as constant  $\vec{F}^2$ ), equation (43) can be interpreted as a Langevin equation of motion for a particle with momentum  $p_q$  (up to a constant term). The force in terms of the physical time  $t$  is just the force needed to move the quark in a ‘dissipative medium’ characterized by the constant friction coefficient  $\mu$ .

It would be interesting to explore the dynamics of the dressed quark in two or three spatial dimensions. However, as we have just remarked, in that case (28) is highly nonlinear not only in the external force (which it was already in one dimension), but also in the velocity of the quark. For this reason, it is unfortunately very difficult to find analytic solutions to it. In the heavy quark (or small force) approximation where (28) linearizes and reduces to the Lorentz-Dirac equation (38), we could of course carry over to our setting the various solutions that have been worked out in the past. For instance, in the first reference of [7], Rohrlich was able to find an analytic solution for a central force problem using the Frenet equations. Starting from this and perturbing the solution it should be possible to obtain (at least numerically) the first correction beyond Lorentz-Dirac predicted by our framework, and deduce for instance the rate of synchrotron radiation. Such a calculation might shed some additional light on the physics behind these extended objects, but is not central for the purposes of our analysis here, so we prefer to leave it for future work.

### 3.2 Nonrelativistic limit

Let us now choose a specific Lorentz frame  $(\vec{x}, t)$ , and restrict attention to motions such that the quark velocity  $\vec{v} \equiv d\vec{x}/dt$ , acceleration  $\vec{a} \equiv d^2\vec{x}/dt^2$ , jerk  $\vec{j} \equiv d^3\vec{x}/dt^3$ , and all higher derivatives, as well as the force  $\vec{F}$  and its rate of change  $d\vec{F}/dt$ , are small in units of the Compton wavelength  $z_m$ . Under such conditions  $d\tau \simeq dt$ , and the spatial components of (27) adopt the linearized form

$$\frac{d^2\vec{x}}{dt^2} - z_m \frac{d^3\vec{x}}{dt^3} + z_m^2 \frac{d^4\vec{x}}{dt^4} - \dots = z_m \vec{F} . \quad (45)$$

Adding to this expression its time derivative multiplied by  $z_m$ , we obtain the simplified form

$$\frac{d^2\vec{x}}{dt^2} = z_m \vec{F} + z_m^2 \frac{d\vec{F}}{dt} , \quad (46)$$

which is the linearized version of (28). We see here very directly that, as explained in Section 2.2, the unusual dependence on the rate of change of the force encodes an infinite number of higher derivatives of the quark trajectory  $\vec{x}(t)$ , derivatives which in turn reflect the extended nature of the quark.

It is also useful to note that in this linearized limit, the string embedding (29) can be rewritten in the form

$$\vec{X}(t, z) = \vec{x}(t_{\text{ret}}) + (z - z_m) \sum_{l=1}^{\infty} (-z_m)^{l-1} \frac{d^l \vec{x}}{dt^l}(t_{\text{ret}}) , \quad t_{\text{ret}} \equiv t - z + z_m , \quad (47)$$

which involves only the quark worldline  $\vec{x}(t)$  and not the force  $\vec{F}(t)$ . If we plug this into the (quadratic version of the) total string energy  $E(t) \equiv - \int dz \Pi_t^t = \int dz (\dot{\vec{X}}^2 + \vec{X}'^2)/2$ , and imitate Mikhailov's procedure [30] (see Section 2.2), the expression naturally splits into

$$E(t) = \int_{-\infty}^t dt_{\text{ret}} \frac{dE_{\text{rad}}}{dt_{\text{ret}}} + E_q(t) ,$$

with

$$E_q(t) = \frac{1}{2} m \left( \sum_{l=1}^{\infty} (-z_m)^{l-1} \frac{d^l \vec{x}}{dt^l}(t) \right)^2$$

and

$$\frac{dE_{\text{rad}}}{dt_{\text{ret}}} = \left( \sum_{l=2}^{\infty} (-z_m)^{l-2} \frac{d^l \vec{x}}{dt^l}(t_{\text{ret}}) \right)^2 + \frac{z_m}{(t - t_{\text{ret}} + 1)^2} \frac{d\vec{x}}{dt}(t_{\text{ret}}) \cdot \left( \sum_{l=2}^{\infty} (-z_m)^{l-2} \frac{d^l \vec{x}}{dt^l}(t_{\text{ret}}) \right) .$$

Using (45), it is easy to check that these expressions indeed coincide with the non-relativistic limit of the time component of (34) and (35). The spatial components can be verified in a similar manner. Notice that, in the relativistic case analyzed in [25] and Section 2.2 of the present work, the explicit presence of the force  $\mathcal{F}^\mu$  in

the solution (29) prevents us from deriving the split (33) as we did here, directly imitating Mikhailov's procedure. It is therefore comforting to see that, at least in the non-relativistic case, the direct result at  $z = z_m$  indeed agrees with what we had previously inferred indirectly from Mikhailov's unambiguous split for the auxiliary data at  $z = 0$ .

Just like in the fully relativistic setting, if we directly substitute (47) into the (quadratic version of the) string action (5) + (6), we *do not* arrive at a variational principle that correctly encodes the equation of motion (45) or (46). Here, however, it is easy to write down an action that does the right job. In fact, we have not one but (at least) two options: we can evidently get (45) from

$$S_{\text{nr}} = -\frac{m}{2} \int dt \left[ \left( \frac{d\vec{x}}{dt} \right)^2 + \left( \frac{d^2\vec{x}}{dt^2} \right)^2 + \left( \frac{d^3\vec{x}}{dt^3} \right)^2 + \dots \right] + \int dt \vec{F} \cdot \vec{x} ,$$

and (46) evidently follows from

$$S'_{\text{nr}} = -\frac{m}{2} \int dt \left( \frac{d\vec{x}}{dt} - z_m \int^t dt' \vec{F}(t') - z_m^2 \vec{F}(t) \right)^2 .$$

It is curious to note that the action  $S_{\text{nr}}$  in terms of higher order derivatives of the position vector with respect to  $\tau$  is reminiscent of the dynamical description of a particle in noncommutative symplectic mechanics given in [57], where it was found that the action can be written in terms of higher order derivatives of the position vector despite the fact that the equations of motion are of second order.

Let us now explore some solutions. Either from  $S'_{\text{nr}}$  or directly from the equation of motion (46), one immediately has a first integral of the motion,

$$\frac{d\vec{x}}{dt} = z_m \int_{-\infty}^t dt' \vec{F}(t') + z_m^2 \vec{F}(t) + \vec{v}_{-\infty} .$$

The case with zero acceleration corresponds to  $\vec{F} + z_m \dot{\vec{F}} = 0$ , whose solution is  $\vec{F}(t) = \vec{F}_0 \exp[-(t - t_0)/z_m]$ . This is of course simply the non-relativistic limit of (41). Since we are working now in the linearized approximation, this solution of the homogeneous equation of motion can be added to any solution of the inhomogeneous equation (46) to yield another solution, so it is very clear that the non-uniqueness of the force is present for any quark trajectory whatsoever. The reason is by now familiar to us: the energy provided to the dressed quark by this one-parameter family of forces has the effect of modifying the gluonic field profile, and consequently does not translate into an increase of velocity.

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